TWO CALCULUS RELATED QUESTIONS

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Problem 1. Draw a right triangle $\triangle A_0B_0C$ with the hypothenuse A_0C of length 1, and the angle $\angle CA_0B_0 = x$, $0 < x < \pi/2$.

For each $i = 0, 1, 2, 3\ldots$ recursively construct a sequence of points $A_i \in \overline{A_0C}$ and $B_i \in \overline{B_0C}$ so that $|\overline{A_{i-1}A_i}| = |\overline{A_{i-1}B_{i-1}}|$ and $\overline{A_iB_i} \perp \overline{B_0C}$.

Label the lengths of line segments: $a_i = |\overline{A_iC}|$, $b_i = |\overline{A_iB_i}|$, and the arc length $c_i = |\overset{\frown}{A_{i+1}B_i}|$. Finally, let d_i be the area of the sector $B_iA_iA_{i+1}$.

See Figure 1.

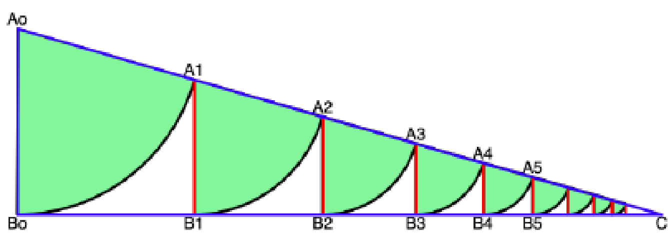

FIGURE 1. $\triangle A_0B_0C$ and points A_i and B_i, $i = 0, 1, 2, 3\ldots$

Find the following four quantities:

a) $a = \sum_{i=0}^{\infty} a_i$
b) $b = \sum_{i=0}^{\infty} b_i$
c) $c = \sum_{i=0}^{\infty} c_i$
d) $d = \sum_{i=0}^{\infty} d_i$

Solution; Observe that $a_0 = 1$, $b_0 = \cos x$, $c_0 = x\cos x$ and $d_0 = \frac{1}{2}x\cos^2 x$. Then

$$\begin{aligned} a_1 &= a_0 - b_0 = 1 - \cos x \\ b_1 &= a_1 \cos x = \cos x(1 - \cos x) \\ c_1 &= xb_1 = x\cos x(1 - \cos x) \\ d_1 &= \frac{1}{2}xb_1^2 = \frac{1}{2}x\cos^2 x(1 - \cos x)^2 \end{aligned}$$

Proceeding by induction, we obtain

$$\begin{aligned} a_i &= a_{i-1} - b_{i-1} = (1-\cos x)^{i-1} - \cos x(1-\cos x)^{i-1} = (1-\cos x)^1 \\ b_i &= a_i \cos x = \cos x(1-\cos x)^i \\ c_i &= xb_i = x\cos x(1-\cos x)^i \\ d_i &= \frac{1}{2}xb_i^2 = \frac{1}{2}x\cos^2 x(1-\cos x)^{2i} \end{aligned}$$

Because $0 < 1 - \cos x < 1$,

$$\begin{aligned} a &= \sum_{i=0}^{\infty} a_i = \sum_{i=0}^{\infty}(1-\cos)^i = \frac{1}{1-(1-\cos x)} = \frac{1}{\cos x} \\ b &= \sum_{i=0}^{\infty} b_i = \cos x \cdot a = 1 \\ c &= \sum_{i=0}^{\infty} c_i = xb = x \end{aligned}$$

and

$$d = \sum_{i=0}^{\infty} d_i = \frac{1}{2}x\cos^2 x \cdot \sum_{i=0}^{\infty}(1-\cos)^{2i} = \frac{\frac{1}{2}x\cos^2 x}{1-(1-\cos x)^2)} = \frac{x\cos x}{2(2-\cos x)}.$$

Problem 2. Consider the integral

$$I_k = \int_0^{\pi/2} x\cos^k x dx, \;\; k = 0, 1, 2, \ldots$$

a) Find a recursive formula for the sequence I_k, $k = 0, 1, 2, \ldots$

b) Use your result from a) to prove that

$$I_k = a_k\pi^2 + b_k\pi + c_k$$

for some rational numbers a_k, b_k, and c_k.

Solution:

a) Start by observing that $(\sin x \cos^{k-1} x)' = k\cos^k x - (k-1)\cos^{k-2} x$ and concluding that

$$\frac{1}{k}x(\sin x\cos^{k-1} x)' = x\cos^k x - \frac{k-1}{k}\cos^{k-2} x.$$

Hence

$$\frac{1}{k}\int_0^{\pi/2} x(\sin x\cos^{k-1} x)'dx = \int_0^{\pi/2} x\cos^k x dx - \frac{k-1}{k}\int_0^{\pi/2}\cos^{k-2} x dx = I_k - \frac{k-1}{k}I_{k-2}.$$

Now, using integration by parts we obtain $\int_0^{\pi/2} x(\sin x\cos^{k-1} x)'dx = -\frac{1}{k}$. Therefore

$$I_k = \frac{k-1}{k}I_{k-2} - \frac{1}{k^2}.$$

b) Note that

$$I_0 = \int_0^{\pi/2} x = \frac{\pi^2}{8} \text{ and } I_1 = \int_0^{\pi/2} x \cos x = -1 + \frac{1}{2\pi}.$$

From a), if k is an even number then

$$I_k = \frac{k-1}{k} I_{k-2} - \frac{1}{k^2} = \frac{k-1}{k} \cdot \left(\frac{k-3}{k-2} I_{k-4} - \frac{1}{(k-2)^2} \right) - \frac{1}{k^2} = \ldots = \alpha_k I_0 + \beta_k = \frac{\alpha_k}{8} \pi^2 + \beta_k$$

for some rational numbers α_k and β_k.

Similarly, if k is odd, for some rational numbers γ_k and δ_k,

$$I_k = \gamma_k I_1 + \delta_k = \frac{\gamma}{2} \pi + \delta_k - 1.$$

Hence, for any $k = 0, 1, 2, \ldots$,

$$I_k = a_k \pi^2 + b_k \pi + c_k$$

for some rational numbers a_k, b_k, and c_k.