1. There are 4 vessels (labelled $1, 2, 3, 4$) containing 36 litres of water in total. Let x_i denote the quantity of water in vessel i, and suppose the water is distributed in such a way that $x_1 > x_2 > x_3 > x_4 \geq 0$. The following procedure is carried out at least once: the contents of the vessel with the most water is distributed evenly among the other 3 vessels. It is observed afterwards that the amounts of water in the vessels is exactly as it was in the start: vessel 1 again has x_1 in it, vessel 2 again has x_2 etc.

   Determine with proof the possible values for x_1.

Solution: We record the states of the vessels at any time as a vector $[y_1\, y_2\, y_3\, y_4]^T$, always arranged so that $y_1 \geq y_2 \geq y_3 \geq y_4$. The effect of the procedure is the mapping

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \mapsto \begin{bmatrix} y_2 + y_1/3 \\ y_3 + y_1/3 \\ y_4 + y_1/3 \\ 0 \end{bmatrix}$$

which is a linear transformation T.

The linear transformation has characteristic polynomial $p(\lambda) = \lambda^4 - \lambda^3/3 - \lambda^2/3 - \lambda/3$. We can factor this as $\lambda(\lambda - 1)(\lambda^2 + 2\lambda/3 + 1/3)$, so the eigenvalues of T are

$$\lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = \frac{-1}{3} + i\frac{\sqrt{2}}{3}, \quad \lambda_4 = \frac{-1}{3} - i\frac{\sqrt{2}}{3}.$$

Observe that the two non-real eigenvalues, λ_3, λ_4 each have magnitude $\sqrt{5}/3$, which is less than 1. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ denote eigenvectors for these eigenvalues in the order listed. We may write any initial state $\mathbf{w}$ as a linear combination

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4$$

Therefore, if $m \geq 1$,

$$T^m\mathbf{w} = c_2\mathbf{v}_2 + c_3\lambda_3^m\mathbf{v}_3 + c_4\lambda_4^m\mathbf{v}_4.$$

By comparing magnitudes, $c_3\lambda_3^m = c_3$ if and only if $c_3 = 0$, and similarly for c_4. Therefore, if $T^m\mathbf{w} = \mathbf{w}$, it must be the case that $\mathbf{w}$ is an eigenvector for the eigenvalue 1. An eigenvector for this eigenvalue may easily be found by solving

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} y_2 + y_1/3 \\ y_3 + y_1/3 \\ y_4 + y_1/3 \\ 0 \end{bmatrix}$$

to get

$$\begin{bmatrix} y_1 \\ 2y_1/3 \\ y_1/3 \\ 0 \end{bmatrix}.$$

So we see that the initial state is of the form $x_1, 2x_1/3, x_1/3, 0$. Since these quantities must add to 36, we must have $x_1 = 18$.

To check that this does indeed give a solution, consider the situation where the vessels contain $18, 12, 6, 0$ litres of water. Once we carry out the procedure 4 times, the situation is back to what it was at the start.

Therefore the possible values for x_1 is 18 and only 18.

2. Recall that the *trace* of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the quantity $\operatorname{Tr}(A) = a + d$. It is also the case that $\operatorname{Tr}(A)$ is the sum of the eigenvalues of A, with multiplicity.

   Suppose A and B are two 2×2-matrices with complex entries such that $\operatorname{Tr}(A) = \operatorname{Tr}(B) = 0$ and
   $$\operatorname{Tr}(A^2)\operatorname{Tr}(B^2) = \operatorname{Tr}(AB).$$
   Prove that there exists a vector $\mathbf{v}$ that is an eigenvector of both A and B.

Solution:

First, assume A is diagonalizable. We may choose a basis so that
$$A = \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}, \quad B = \begin{bmatrix} b & c \\ d & -b \end{bmatrix}.$$
Now we calculate $\operatorname{Tr}(A^2) = 2a^2$, $\operatorname{Tr}(B^2) = 2(b^2 + cd)$ and
$$AB = \begin{bmatrix} ab & ac \\ -ad & ab \end{bmatrix}.$$
Therefore $\operatorname{Tr}(AB) = 2ab$. We see that $\operatorname{Tr}(A^2)\operatorname{Tr}(B^2) = \operatorname{Tr}(AB)^2$ if
$$4a^2(b^2 + cd) = 4a^2b^2$$
which is to say, if $a^2cd = 0$. If $a = 0$, then $A = 0$ and every eigenvector of B is an eigenvector of A. If $c = 0$, then both A and B are lower triangular, and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector of each. If $d = 0$, then they are upper triangular and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector of each. Therefore if A is diagonalizable, the result is proved.

Now suppose A is not diagonalizable. It must have a repeated eigenvalue, and that repeated eigenvalue must be $0 = \frac{1}{2}\operatorname{Tr}(A)$. We may choose coordinates so that A takes the form
$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$
Write $B = \begin{bmatrix} b & c \\ d & -b \end{bmatrix}$, as before. We calculate $\operatorname{Tr}(A^2) = 0$, and $\operatorname{Tr}(AB) = d$. Therefore $\operatorname{Tr}(A^2)\operatorname{Tr}(B^2) = \operatorname{Tr}(AB)$ if and only if $d = 0$, whereupon A and B are both upper triangular and have $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as a shared eigenvector.